A Brief Introduction to Category Theory for Systems and Software Engineers

Part A: Foundations for Engineering

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Category Theory and Applications

Context and concepts for engineering practice

- Applied category theory has been receiving increased attention e.g.,
  - US National Aeronautics and Space Administration (NASA) [1]
  - The International Conferences on Applied Category Theory (ACT) [2]
  - Massachusetts Institute of Technology – Fong & Spivak [3]
- *Context*: a brief history of category theory in mathematics
- *Concepts*: structures, composition, functors, systems of structures
- *Application to engineering*: the ABCs of engineering problem solving

The purpose of this presentation is to inform nonspecialists at OMG about key concepts and applications of category theory.

Brief History of Category Theory in Mathematics
What is it? Why was it useful?

In general, a *category* is a class of *objects* with shared characteristics [4].

In mathematics, *category theory* is a general theory of algebraic structures and systems of structures [5, §1.1]. It evolved from a 1945 paper by Eilenberg and MacLane, "General Theory of Natural Equivalences" [6].

An early benefit was to show how one *knowledge domain* in mathematics could be associated with another, for example …

A characteristic algebraic invariant such as the genus number* can be associated with a geometrical surface (in topology) such as a sphere or torus.

The genus number of a sphere (e.g., a balloon) in three dimensions is *zero* whereas the number for a torus is *one* (e.g., doughnut ≅ coffee cup in topology).

This is very useful in higher dimensions, where surfaces are not easily visualised.

*Intuitively, the (topological) *genus number* of a connected orientable surface is the number of holes in the surface.

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Brief History of Category Theory in Mathematics
How has it been used in science and engineering?

In 1993 Rosen was perhaps first to note its relevance to the biological sciences [7]. Baez continues this type of investigation today [2; ACT 2023, paper 31].

As a powerful language, or conceptual framework, it has come to occupy a central position in contemporary mathematics and theoretical computer science [5, §1.1].

Several authors have noted its potential relevance as a foundation or framework for systems engineering [8-10], or conceptual applications as in Spivak [3] ***but***,

In engineering research, it tends to focus on *gaining structural insights*.

Breiner et al [10], and others note that application of the theory requires specific and advanced mathematical skills, typically not possessed by engineers.

To date, category theory has provided little impact on *engineering practice*.

Should OMG be Interested in Category Theory?

What you need to know: a relation with object orientation

Category theory reveals how different types of structures are related to one another [5, §1.3].

Key ideas:

Models are central to systems and software

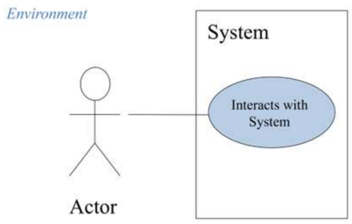
Mathematical models are realised in

relational structures to include

graphical structures (e.g., object orientation)

Predicates can express *relationships between objects*.

In category theory, relationships between two objects are represented by *arrows* called *morphisms*.



(a) Predicate (verb-noun phrase) in Use Case diagram

A category is characterized by its morphisms; *not by its objects*.

$$E \xrightarrow{P} S$$

(b) Predicate as a morphism in the Category of Relations (*Rel*)

Rel : the category of relations. The objects of *Rel* are sets. Its morphisms are binary relations.

Should OMG be Interested in Category Theory?

What you need to know: relation to knowledge representation

Structures in category theory can be used to represent knowledge (e.g., ontologies, models).

Key ideas:

Relations are used to express knowledge about objects.

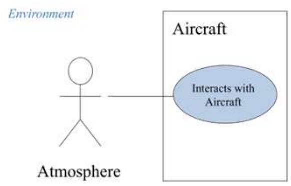
Examples of properties as morphisms in the category *Rel,*

(a) $f: A \rightarrow B$ (barometric pressure vs altitude)

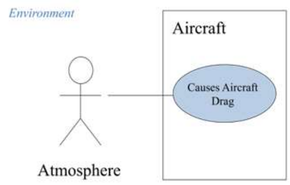
(b) $g: B \rightarrow C$ (drag coefficient vs barometric pressure)

These two *morphisms are abstract structural elements* that reveal how meteorology is related to aerodynamics.

Category theory in mathematics is concerned with *classes of structures of a similar type (i.e., homomorphic). [5, §1.3]**



(a) Predicates and classes of objects can express domain knowledge such as *atmosphere interacts with aircraft.*

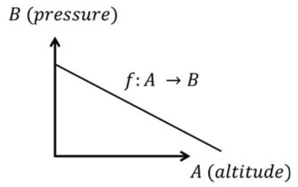


(b) The predicate can be refined to express a property: *atmosphere causes aircraft drag.*

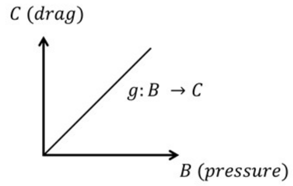
**In [11], Architecture is investigated as a mathematical class and properties of structures.*

The Big Ideas of Category Theory
Structures and Knowledge Representation (1 of 2)

The class of all sets and functions (maps) between pairs of sets is a category denoted as *Set**. Because all its objects are *sets*, it is called a *concrete category*.



In science and engineering, properties of the objects of *Set* are expressed through morphisms (maps) between sets such as $f: A \to B$ and $g: B \to C$. These maps can be modelled in SysMLv2 as constraints.



In category theory, the internal details of the morphisms (maps) are supressed. This is like the *doughnut and coffee cup in topology*.

Category theory in mathematics views these two graphs as one type of structure.

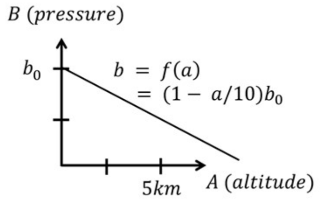
*An alternative notation is **Set**. The category *Set* is a subclass of the category *Rel*. Its morphisms are functions.

The Big Ideas of Category Theory
Structures and Knowledge Representation (2 of 2)

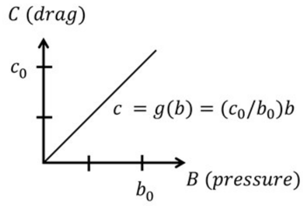
Science and engineering typically involve exposing the 'internal details' of the objects and morphisms.

For the meteorology example, the map $f: A \to B$ can be attributed a formula on the domain $A = [0, 5]$.

So too the map $g: B \to C$ for drag in aerodynamics can be attributed a domain $B = [0, b_0]$ and formula.



Without endowing the diagrams of category theory with domain knowledge, they are mostly a subject of mathematics and not engineering practice [12].



In engineering analysis, internal details must be exposed but in the mathematics of category theory, they are supressed.

Refer to the Annex for a derivation of the details of the linear approximations in the graphs.

The Big Ideas of Category Theory
Compositions and Structures

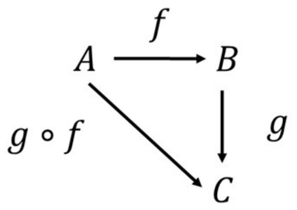
The composition of g with f is the map $g \circ f: A \to C$ defined by the formula, $c = g \circ f(a) = g(f(a))$.

The composition for the drag problem in the category *Set* represents *the dependence of drag on altitude*.

$$b = f(a) = (1 - a/10)b_0$$
$$c = g(b) = \left(\frac{c_0}{b_0}\right) b$$

and

$$c = g \circ f(a) = \left(\frac{c_0}{b_0}\right) b = \left(\frac{c_0}{b_0}\right)(1 - a/10)b_0$$
$$= (1 - a/10)c_0$$



Graphical notation in the category *Set* for composition.

Although the composition in these equations is for knowledge about the drag problem, similar maps can also be defined for *messages and structured exchanges of information* between systems [13].

In category theory, compositions of morphisms form structures.

ABCs of Problem Solving in Category Theory
Modeling, Analysis and Design using Structures

Transform the Use Case predicate into a morphism P in *Rel*.

Express domain knowledge models in *Set* to quantify P as a property.

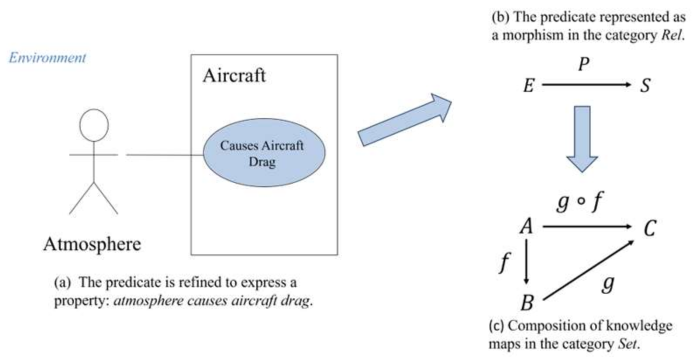
Mathematical models are needed,

$f: A \to B$

$g: B \to C$

$c = g \circ f(a) = (1 - a/10)c_0$

These support *analysis and design*.



Model specification and transformation in *Rel*.

Thus, $c = g \circ f(a)$ is valid on the domain [0, 5 km]. Drag, c, *is minimised* at $a = 5$ km.

A Critical Analysis of Category Theory for Engineering

"I asked for the time of day, and you built me a clock!"

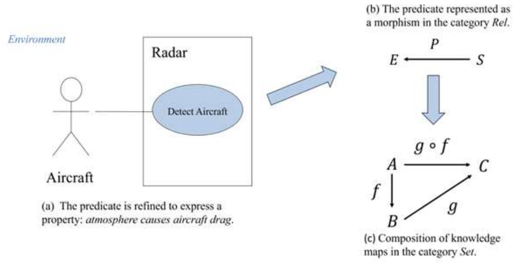
An engineer might say that this is a lot of work to answer a simple question about minimising drag.

One power of category theory [3] is *co-design of components* of a complex system such as a radar.

Compositions provide a rigorous method for reducing complexity:

Multiple design factors can be expressed as single compositions of properties of structural elements across domains.

But in mathematics, the details of properties are supressed in category theory.



Analysis for *co-design of a radar transmitter and antenna*, using power-aperture product. The diagrams have the *same type of structure* as the drag problem but different details: $a \in A$ is the radar range, $b \in B$ is power density (at range a), and $c \in C$ is the power received by an aircraft (at range a). Refer to [12] for further details.

Category theory is a useful *language of architecture* for system modeling & analysis [12].

The Big Ideas of Category Theory

Functors and Systems of Structures

A *functor* is a morphism between categories; informally, a structure preserving map $\mathfrak{F}$ between categories [5, §1.2].

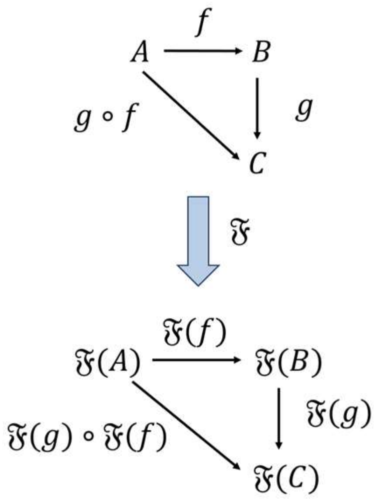
Key idea:

A functor maps objects to objects and morphisms to morphisms in a way that preserves compositions.

A key example is the *power set functor* $\mathcal{P}$ on *Set* that maps each set in *Set* to the set of all its subsets e.g.,

Each morphism $f: A \to B$ is assigned to $V \to f(V)$, where V is a subset of A and $f(V)$ is its image in B.

This is useful for mapping ranges of values such as the altitudes $V = [0, 5\text{km}] \to$ the drag coefficients $\left[\left(\frac{1}{2}\right)c_0, c_0\right]$.



The Big Ideas of Category Theory
Systems of Structures vis-à-vis Categories

A category is characterized by its morphisms (relations); not by its objects (classes).

Engineering is concerned with structured sets related by structure-preserving maps.
These are expressed in Concrete Categories (where the objects are sets).
Category theory treats the notion of structure in a uniform manner.
Almost every known mathematical structure or system of structures with the appropriate structure-preserving map yields a category [5, §1.2].

But not all categories are made of structured sets with structure-preserving maps.*

Informally category theory can be conceived as patterns of relations and interrelations.

*This can be a subject of future presentations, if there is interest.

Summary and Conclusions
What is Category Theory and Why Should OMG Care?

Category theory
Offers a language and theory of algebraic structures and systems of structures.
Is concerned with classes of structures and morphisms between categories.
Can be a *useful language of architecture* for system modeling and analysis.
It promises to provide structural insights into engineering concepts …
But must be complemented with domain knowledge for *engineering practice*.
There are correspondences of morphisms with predicates and constraints in SysML v2.

Way ahead:
Elaborate June presentation on modeling artificial general intelligence (AGI)
Category theoretic foundation of object orientation and software defect rates
Applications to OMG languages and standards e.g., SysMLv2, ontologies, …

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Annex

- Genus of a Connected Orientable Surface
- Derivation of the Meteorological Domain Model
- Derivation of the Aerodynamics Domain Model

Genus of a Connected Orientable Surface

In topology, the genus of a connected orientable surface is an integer representing the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected.

A Klein bottle is a closed, single-sided mathematical surface of genus 2, sometimes described as a closed bottle for which there is no distinction between inside and outside.

A genus is a class or group of something. In biology, it's a taxonomic group covering more than one species. This is a term used by biologists to classify more than one species under a larger umbrella. In biology, the word *family* describes the broadest group category, then *genus*, and then *species*.

Derivation of the Meteorological Domain Model

In science and engineering, the representation of domain knowledge typically involves exposing the 'internal details' of the objects and morphisms (maps).

For the meteorology example, the map f : *A* → *B* can be attributed a formula,

$b = f(a) = (1 - a/10))b_0$; a is altitude and b is barometric pressure

Here, b_0 is the standard atmosphere barometric pressure and a is in km.

The formula is a linear approximation. Its *domain A* is from a = 0 km (sea level) to a = 5 km, and its *codomain B* is from $b = b_0$ to (1/2)b_0. This model represents the knowledge that b is known to decrease about 50% from its sea level value because about half of the molecules of the earth's atmosphere are below 5 km, altitude.

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Derivation of the Aerodynamics Domain Model

For the aerodynamics example, the map *g*: *B* → *C* can be attributed a formula,

$c = g(b) = (c_0/b_0))b$; $c_0$is (pressure) drag coefficient at sea level.

Here, b_0 is the standard atmosphere barometric pressure and a is in km.

The formula is a linear approximation derived from the ideal gas law. Its *domain B* is from $b = b_0$ to (1/2)b_0. The codomain *C* is from c_0 to (1/2)c_0.

This range of values of c is determined by evaluating $g(b)$ at b_0 and (1/2)b_0. This is an example of what is called *composition of maps* in mathematics. Specifically, $b_0 = f(0)$ and (1/2)$b_0 = f(5)$. *Composition is central to category theory.*